# Implicitization of curves and surfaces using predicted support 

Ioannis Z. Emiris ${ }^{a}$, Tatjana Kalinka ${ }^{a}$, Christos Konaxis ${ }^{b}$, Thang Luu $\mathrm{Ba}^{a, c *}$<br>${ }^{a}$ Department of Informatics \& Telecommunications, University of Athens, Greece<br>${ }^{b}$ Archimedes Center for Modeling, Analysis \& Computation (ACMAC), University of Crete, Heraklio, Greece<br>${ }^{c}$ Department of Mathematics, Hanoi National University of Education, Hanoi, Vietnam

December 15, 2011


#### Abstract

We reduce implicitization of rational planar parametric curves and (hyper)surfaces to linear algebra, by interpolating the coefficients of the implicit equation. For this, one may use any method for predicting the implicit support. We focus on methods that exploit input structure in the sense of sparse (or toric) elimination theory, namely by computing the Newton polytope of the implicit polynomial, via sparse resultant theory. We implement our methods on Maple, and some on Matlab as well, and study their numerical stability and efficiency on several classes of curves and surfaces. We apply our approach to approximate implicitization, and quantify the accuracy of the approximate output, which turns out to be satisfactory on all tested examples. In building a square or rectangular matrix, an important issue is (over)sampling the given curve or surface: we conclude that unitary complexes offer the best tradeoff between speed and accuracy when numerical methods are employed, namely SVD, whereas for exact kernel computation random integers is the method of choice.


## 1 Introduction

Implicitization is the problem of changing the representation of parametric objects to implicit form. It lies at the heart of several questions in computer-aided geometric design (CAGD) and geometric modeling, including intersection problems and membership queries. In several situations, it is important to have both representations available. Implicit representations encompass a larger class of shapes than parametric ones. Moreover, the class of implicit curves and surfaces is closed under certain operations such as offsetting, while its parametric counterpart is not. Implicitization is also of independent interest, since certain questions in areas as diverse as robotics or statistics, e.g. CTY10, reduce to deriving the implicit form.

Here we follow a classical symbolic-numeric method, which reduces implicitization to interpolating the coefficients of the defining equation. We implement interpolation by exact or numeric linear algebra, following a symbolic phase of implicit support prediction, which computes a (super)set of the monomials appearing in the implicit equation. Standard methods to determine the unknown coefficients by linear algebra are divided in two main categories, dense and sparse methods. The former require only a bound on the total degree of the target polynomial, whereas the latter require a bound on the number of its terms, thus exploiting any sparseness of the target polynomial. A priori knowledge of the support helps significantly, by essentially answering the first step of sparse interpolation algorithms.

One contribution of this paper is to exploit sparse (or toric) variable elimination theory to predict the implicit Newton polytope, i.e. the convex hull of the implicit support.

Definition 1. Given a polynomial

$$
\sum_{a \in A_{i}} c_{i a} t^{a} \in \mathbb{R}\left[t_{1}, \ldots, t_{n}\right], \quad t^{a}=t_{1}^{a_{1}} \cdots t_{n}^{a_{n}}, a \in \mathbb{N}^{n}, c_{i a} \in \mathbb{R}-\{0\}
$$

its support is the set $A_{i}=\left\{a \in \mathbb{N}^{n}: c_{i a} \neq 0\right\}$; its Newton polytope is the convex hull of the support.
One reason for revisiting interpolation of the implicit coefficients is the current increase of activity around various approaches capable of predicting the implicit support. Our team has been focusing on sparse elimination theory EFKP11, EKP10, EK03, EKK11. Recent support prediction methods notably include tropical geometry methods, e.g. Cue10, DS10, JY11, STY07, SY08. The present work relies on the implicit support

[^0]

Figure 1: Examples of Newton polygons $N\left(f_{i}\right)$ of polynomials $f_{i} \in \mathbb{Z}[x, y]$.
predicted by any method. In fact, SY08, sec.4] states that "Knowing the Newton polytopes reduces computing the [implicit] equation to numerical linear algebra. The numerical mathematics of this problem is interesting and challenging [..] " Our implementations interface linear algebra interpolation with the implicit support predictors of EFKP11, EKP10, but we expect to also interface predictors from [Cue10, EFKP11. In the sequel, we juxtapose the use of exact and numerical linear algebra.

In practical applications of CAGD, precise implicitization often can be impossible or very expensive to obtain. Approximate implicitization over floating-point numbers appears to be an effective solution DT03, SJ08, BD10a, BD10b. We discuss approximate implicitization, in the setting of sparse elimination. Approximate implicitization is one of the main motivations for reducing implicitization to interpolation of the implicit coefficients. We offer a Maple implementation, available upon request from the authors, based on our software for computing implicit polytopes [EFKP11]. The latter is also available as a C++ implementation ${ }^{1}$ We study the numerical stability and efficiency of our algorithms on several classes of curves and surfaces. One central question is how to evaluate the computed monomials to obtain a suitable matrix, when performing exact or numerical matrix operations. We compare results obtained by using random integers, random complex unitary numbers and complex roots of unity. It appears that complex unitary numbers offer the best tradeoff of efficiency and accuracy for numerical computation, whereas random integers are preferred for exact kernel computation.

Let us now define the problem formally. A parametrization of a geometric object of co-dimension one, in a space of dimension $n+1$, can be described by parametric map:

$$
f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1}: t=\left(t_{1}, \ldots, t_{n}\right) \mapsto x=\left(x_{0}, \ldots, x_{n}\right)
$$

where $t$ is the vector of parameters and $f:=\left(f_{0}, \ldots, f_{n}\right)$ is a vector of continuous functions, including polynomial, rational, and trigonometric functions, also called coordinate functions. These are defined on some product of intervals $\Omega:=\Omega_{1} \times \cdots \times \Omega_{n}, \Omega_{i} \subseteq \mathbb{R}^{n}$. In the case of trigonometric input, it may be converted to polynomials by the standard half-angle transformation

$$
\sin \theta=\frac{2 \tan \theta / 2}{1+\tan ^{2} \theta / 2}, \cos \theta=\frac{1-\tan ^{2} \theta / 2}{1+\tan ^{2} \theta / 2}
$$

where the parametric variable becomes $t=\tan \theta / 2$.
The implicitization problem asks for the smallest algebraic variety containing the closure of the image of the parametric map $f: t \mapsto f(t)$. This image is contained in the variety defined by the ideal of all polynomials $p\left(x_{0}, \ldots, x_{n}\right)$ s.t. $p\left(f_{0}(t), \ldots, f_{n}(t)\right)=0$, for all $t$ in $\Omega$. We restrict ourselves to the case when this is a principal ideal, and we wish to compute its defining polynomial

$$
\begin{equation*}
p\left(x_{0}, \ldots, x_{n}\right)=0 \tag{1}
\end{equation*}
$$

given its Newton polytope, which we call implicit polytope, or a polytope that contains the implicit polytope. If the degree of the parametrization is $\theta>1$, our method computes the implicit equation to the power $\theta$.

We can regard the variety in question as the projection of the graph of map $f$ to the last $n+1$ coordinates. If $f$ is polynomial, implicitization is reduced to eliminating $t$ from the polynomial system

$$
\bar{F}_{i}:=x_{i}-f_{i}(t) \in\left(\mathbb{R}\left[x_{i}\right]\right)[t], i=0, \ldots, n,
$$

seen as polynomials in $t$ with coefficients which are functions of the $x_{i}$. This is also the case for rational parameterizations

$$
\begin{equation*}
x_{i}=\frac{f_{i}(t)}{g_{i}(t)}, i=0, \ldots, n \tag{2}
\end{equation*}
$$

[^1]which can be represented as polynomials
\[

$$
\begin{equation*}
\bar{F}_{i}:=x_{i} g_{i}(t)-f_{i}(t) \in\left(\mathbb{R}\left[x_{i}\right]\right)[t], i=0, \ldots, n, \tag{3}
\end{equation*}
$$

\]

where we have to take into account that the $g_{i}(t)$ cannot vanish by adding the polynomial

$$
\begin{equation*}
\bar{F}_{n+1}=1-g_{0}(t) \cdots g_{n}(t) y, \tag{4}
\end{equation*}
$$

where $y$ is a new variable.
Several algorithms exist for implicitization, including methods based on resultants, Gröbner bases, $\mu$-bases and moving surfaces, and residues. Our approach relies on any support prediction method, see sect. 3. We focus on sparse (or toric) elimination: In the case of curves, the implicit support is directly determined for generic parametric expressions with the same supports [EKP10. In the general case, the implicit support is provided by that of a sparse resultant whose Newton polytope is projected to the space of the $x_{i}$ 's EFKP11]. Our approach can handle $f_{i}$ with (certain) symbolic nonzero coefficients, thus computing the implicit polytope for entire families of parametric objects, when used with exact solving.

The rest of the paper is structured as follows. Previous work is discussed in the next two sections: Section 2 discusses existing methods for interpolating the implicit polynomial's coefficients by linear algebra. Section 3 discusses implicit support prediction and sparse elimination theory. Our algorithm is detailed in section 4 , where we discuss complexity issues, present some examples, and mention possible algorithmic extensions. The algorithm's implementation, performance, and numerical accuracy in the case of approximate implicitization are described in section 5. We conclude with future work in section 6. The Appendix contains examples of exact and approximate implicit equations of parametric curves and surfaces used in our experiments, and further experimental results.

A preliminary version of partial results from this paper appeared as EKK11.

## 2 Existing interpolation methods

This section examines how implicitization had been reduced to interpolation. Throughout the paper, we use interpolation to refer to the method of determining the implicit coefficients from the implicit support and its evaluations on points of our choice. Of course, these points lie in the space of parameters.

Let $S$ be (a superset of) the support of the implicit polynomial $p\left(x_{0}, \ldots, x_{n}\right)=0, \vec{p}$ be the $|S| \times 1$ vector of its unknown coefficients, and let $m=|S|$. We refer to $S$ as implicit support, with the understanding that it may be a superset of the actual support.

Sparse interpolation is the problem of interpolating a multivariate polynomial when information of its support is given Zip93. This may simply be a bound $\sigma$ on support cardinality, then sparse interpolation is achieved in $O\left(m^{3} \delta n \log n+\sigma^{3}\right)$, where $\delta$ bounds the output degree per variable, $m$ is the actual support cardinality, and $n$ the number of variables BOT88, KL89. A probabilistic approach runs in $O\left(m^{2} \delta n\right)$ Zip90 and requires as input only $\delta$.

For the sparse interpolation of resultants, the quasi-Toeplitz structure of the matrix allows us to reduce complexity by one order of magnitude, when ignoring polylogarithmic factors, and arrive at a quadratic complexity in matrix size CKL89. This was extended to the case of sparse resultant matrices EP02, EP05.

Our matrices reveal what we call quasi-Vandermonde structure, since the matrix columns are indexed by monomials and the rows by values on which the monomials are evaluated. This reduces matrix-vector multiplication to multipoint evaluation of a multivariate polynomial. It is unclear how to achieve this post-multiplication in time quasi-linear in the size of the polynomial support when the evaluation points are arbitrary, as in our case. Existing work achieves quasi-linear complexity for specific points EP02, Pan94, Sau04, vdHS10].

### 2.1 Exact implicitization

The most direct method to reduce implicitization to linear algebra is to construct a $|S| \times|S|$ matrix $M$, indexed by monomials with exponents in $S$ (columns) and $|S|$ different values (rows) at which all monomials get evaluated. Then, vector $p$ is in the kernel of $M$. This idea was used in EK03, MM02, SY08; it is the approach explored in this paper, extended to an approximate implicitization as well.

In STY07, they propose evaluation at unitary $\tau \in\left(\mathbb{C}^{*}\right)^{n}$, i.e., of modulus 1 . This is one of the evaluation strategies examined below. Another approach was described in CGKW00, based on integration of matrix $M=S S^{T}$, over each parameter $t_{1}, \ldots, t_{n}$. Then, $p$ is in the kernel of $M$. In fact, the authors propose to consider successively larger supports in order to capture sparseness. This method covers a wide class of parameterizations, including polynomial, rational, and trigonometric representations, but the size of $M$ is quite big and matrix entries take big values, so it is difficult to control its numeric corank. In some cases, its corank is $\geq 2$. Thus, the accuracy, or quality, of the approximate implicit polynomial is unsatisfactory. The resulting
matrix has Henkel-like structure KL03. When it is computed over floating-point numbers, the resulting implicit polynomial does not necessarily have integer coefficients. In CGKW00, they discuss some post-processing to yield the integer relations among the coefficients, but only for small examples.

### 2.2 Approximate implicitization

Approximate implicitization over floating-point numbers was introduced by T. Dokken and co-workers in a series of papers. Today, there are direct [DT03, WTJD04 and iterative techniques APJ11. We describe the basic direct method DT03: Given a parametric (spline) curve or surface $x(t), t \in \Omega \subset \mathbb{R}^{n}$, the goal is to find polynomial $q(x)$ such that $q(x(t)+\eta(t) g(t))=0$, where $g(t)$ is a continuous direction function with Euclidean norm $\|g(t)\|=1$ and $\eta(t)$ a continuous error function with $|\eta(t)| \leq \epsilon$. Now, $q(x(t))=(M p)^{T} \alpha(t)$, where matrix $M$ is built from monomials in $x$. It may be constructed as in this paper, or it may contain a subset of the monomials of the implicit support. Moreover, $p$ is the vector of implicit coefficients, hence $M p=0$ returns the exact solution, and $\alpha(t)$ is the basis of the space of polynomials which describes $q(x(t))$, and is assumed to form a partition of unity and to be nonnegative over $\Omega$ : $\sum_{i} \alpha_{i}=1, \alpha_{i} \geq 0, \forall i, t \in \Omega$. One may use the Bernstein-Bézier basis with respect to $\Omega$, in the case of curves, or a triangle which contains $\Omega$, in the case of surfaces.

In [DT03, p.176] the authors propose to translate to the origin and scale the parametric object, so as to lie in $[-1,1]^{n}$, in order to improve the numerical stability of the linear algebra operations. In our experiments, we found out that using unitary complex values leads to better numerical stability. Since both our and their methods rely on SVD, our experiments confirm their findings.

The idea of the above methods is to interpolate the coefficients using successively larger supports, starting with a quite small support and extending it so as to reach the exact one. Existing approaches have used upper bounds on the total implicit degree, thus ignoring any sparseness structure. Our methods provide a formal manner to examine different supports, in addition to exploiting sparseness; we are currently investigating this idea.

In the context of sparse elimination, the Newton polytope captures the notion of degree. Given an implicit polytope we can naturally define candidates of smaller support, the equivalent of lower degree in classical elimination, by an inner integral offset of the implicit polytope. The offset operation can be repeated, thus producing a list of implicit supports yielding smaller implicit equations with larger approximation error. Clearly, the computed implicit equation is of lower degree than the actual one. It is an open question to bound the difference between the input geometric object and the zero-set, or variety, defined by the approximate equation. The specific algorithm and software used to compute the implicit support, namely [EFKP11], should offer the possibility of a faster approximate computation of the implicit polytope. This is particularly relevant in the present context and we plan to further investigate it when completed.

## 3 Support prediction

This section describes our methods for computing the implicit support, which is based on the sparse resultant, and any information we obtain from this computation towards computing the implicit equation.

Most existing approaches for support prediction employ total degree bounds on the implicit polynomial to compute a superset of the implicit support, e.g. CGKW00. This fails to take advantage of the sparseness of the input in order to accelerate computation, and to exploit sparseness in the implicit polynomial in the sense of prop. 6. For this, computing the Newton polytope of a rational hypersurface was posed in [SY94] for generic Laurent polynomial parameterizations, in the framework of sparse elimination theory.

Algorithms based on tropical geometry have been offered in DFS07, STY07, SY08. This method computes the abstract tropical variety of a hypersurface parameterized by generic Laurent polynomials in any number of variables, thus yielding its implicit support; it is implemented in TrIm. For non-generic parameterizations of rational curves, the implicit polygon is predicted. In higher dimensions, the following holds:

Proposition 1. [STY07, prop.5.3] Let $f_{0}, \ldots, f_{n} \in \mathbb{C}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$ be any Laurent polynomials whose ideal of algebraic relations is principal, say $I=\langle g\rangle$, and $P_{i} \subset \mathbb{R}^{n}$ the Newton polytope of $f_{i}$. Then, the polytope constructed combinatorially from $P_{0}, \ldots, P_{n}$ using tropical geometry contains a translate of the Newton polytope of $g$.

The tropical approach was improved in Cue10 to yield the precise implicit polytope in $\mathbb{R}^{3}$ for generic parameterizations of surfaces in 3 -space. In JY11, they describe efficient algorithms implemented in the GFan library Jen10 for the computation of Newton polytopes of specialized resultants, which may then be applied to predict the implicit polytope. Sparse elimination has been used for the same task [EFKP11, as discussed below. The latter is faster on dimensions relevant here, namely for projected polytopes in up to 5 dimensions.

The Newton polygon of a curve parameterized by rational functions, without any genericity assumption, is determined in [DS10. In a similar direction, an important connection with combinatorics was described in EK06, as they showed that the Newton polytope of the projection of a generic complete intersection is isomorphic to the mixed fiber polytope of the Newton polytopes associated to the input data.

In EKP10, sparse elimination is applied to determine the vertex representation of the implicit Newton polygon of planar curves. The method uses mixed subdivisions of the input Newton polygons and regular triangulations of pointsets defined by the Cayley trick. It can be applied to polynomial and rational parameterizations, where the latter may have the same or different denominators. The method offers a set of rules that, applied to the supports of sufficiently generic rational parametric curves, specify the 4,5 , or 6 vertices of the implicit polygon. In case of non-generic inputs, this polygon is guaranteed to contain the Newton polygon of the implicit equation. The method can be seen as a special case of the general approach based on sparse elimination.

In EK03] a method relying on sparse elimination for computing a superset of the generic support from the resultant polytope is discussed, itself obtained as a (non orthogonal) projection of the secondary polytope. The latter was computed by calling Topcom Ram02. This approach was quite expensive and, hence, applicable only to small examples; it is refined and improved in this paper.

### 3.1 Sparse elimination theory

Sparse, or toric, elimination subsumes classical, or dense, elimination in the sense that, when Newton polytopes equal the corresponding simplices, the former bounds become those of the classical theory [EK03, sec.3], [SY94, thm.2(2)].

Consider the polynomial system $\bar{F}_{0}, \ldots, \bar{F}_{n} \in K[t]$ as in (3), defining a hypersurface, and let $A_{i} \subset \mathbb{Z}^{n}$ be the support of $\bar{F}_{i}$ and $P_{i} \subset \mathbb{R}^{n}$ the corresponding Newton polytope. The family $A_{0}, \ldots, A_{n}$ is essential if they jointly affinely span $\mathbb{Z}^{n}$ and every subset of cardinality $j, 1 \leq j<n$, spans a space of dimension $\geq j$. It is straightforward to check this property algorithmically and, if it does not hold, to find an essential subset. In the sequel, the input $A_{0}, \ldots, A_{n} \subset \mathbb{Z}^{n}$ is supposed to be essential.

For simplicity, in what follows we do not consider the extra polynomial in (4) as part of our polynomial systems. This is equivalent to considering polynomial parameterizations. However, it is straightforward to generalize the discussion below to the rational case.

For each $\bar{F}_{i}, i=0, \ldots, n$, we define a polynomial $F_{i} \in K[t]$ with symbolic coefficients $c_{i j}$ and the same support $A_{i}$, i.e. a generic polynomial with respect to $A_{i}$ :

$$
\begin{equation*}
F_{i}=\sum_{j=1}^{\left|A_{i}\right|} c_{i j} t^{a_{i j}} \in K[t], a_{i j} \in A_{i}, i=0, \ldots, n \tag{5}
\end{equation*}
$$

Obviously, each $F_{i}$ has also the same Newton polytope $P_{i}$ as $\bar{F}_{i}$.
Now we introduce our main tool, namely the resultant of an overconstrained polynomial system. The resultant of polynomial system (5) is an irreducible polynomial

$$
\mathcal{R} \in \mathbb{Z}\left[c_{i j}: i=0, \ldots, n, j=1, \ldots,\left|A_{i}\right|\right]
$$

defined up to sign, vanishing iff $F_{0}=F_{1}=\cdots=F_{n}=0$ has a common root in a specific variety: This variety is the projective variety $\mathbb{P}^{n}$ over the algebraic closure $\bar{K}$ of $K$, in the case of projective (or classical) resultants, or the toric variety $X$ defined by the $A_{i}$ 's in the case of sparse (or toric) resultants; $X$ contains the topological torus as a dense subset: $\left(\bar{K}^{*}\right)^{n} \subset X$, and is itself a projective variety in a space of dimension much larger than $n$. The Newton polytope $N(\mathcal{R})$ of the resultant polynomial is the resultant polytope. We call any monomial which corresponds to a vertex of $N(\mathcal{R})$ an extreme term of $\mathcal{R}$.

The Minkowski sum $A+B$ of convex polytopes $A, B \subset \mathbb{R}^{n}$ is the set $A+B=\{a+b \mid a \in A, b \in B\} \subset \mathbb{R}^{n}$. A tight mixed subdivision of $P=P_{0}+\cdots+P_{n}$, is a collection of $n$-dimensional convex polytopes $\sigma$, called (Minkowski) cells, s.t.: They form a polyhedral complex that partitions $P$, and every cell $\sigma$ is a Minkowski sum of subsets $\sigma_{i} \subset P_{i}: \sigma=\sigma_{0}+\cdots+\sigma_{n}$, where $\operatorname{dim}(\sigma)=\operatorname{dim}\left(\sigma_{0}\right)+\cdots+\operatorname{dim}\left(\sigma_{n}\right)=n$.

A cell $\sigma$ is called $v_{i}$-mixed if it is the Minkowski sum of $n$ one-dimensional segments $E_{j} \subset P_{j}$ and one vertex $v_{i} \in P_{i}: \sigma=E_{0}+\cdots+v_{i}+\cdots+E_{n}$. A mixed subdivision is called regular if it is obtained as the projection of the lower hull of the Minkowski sum of lifted polytopes $\widehat{P}_{i}:=\left\{\left(p_{i}, \omega\left(p_{i}\right)\right) \mid p_{i} \in P_{i}\right\}$. If the lifting function $\omega$ is sufficiently generic, then the induced mixed subdivision is tight.

The mixed volume of $n$ polytopes in $\mathbb{R}^{n}$ equals the sum of the volumes of all the mixed cells in a mixed subdivision of their Minkowski sum. We recall a surjection from the regular tight mixed subdivisions to the vertices of the resultant polytope:

Theorem 2. Stu94 Given a polynomial system as in (5) and a regular tight mixed subdivision of the Minkowski sum $P=P_{0}+\cdots+P_{n}$ of the Newton polytopes of the system polynomials, an extreme term of the resultant $\mathcal{R}$ equals

$$
c \cdot \prod_{i=0}^{n} \prod_{\sigma} c_{i \sigma_{i}}^{\operatorname{vol}(\sigma)}
$$

where $\sigma=\sigma_{0}+\sigma_{1}+\cdots+\sigma_{n}$ ranges over all $\sigma_{i}$-mixed cells, and $c \in\{-1,+1\}$.
Computing all regular tight mixed subdivisions reduces, due to the so-called Cayley trick, to computing all regular triangulations of a point set of cardinality $\left|A_{0}\right|+\cdots+\left|A_{n}\right|$ in dimension $2 n$. Let the Cayley embedding of the $A_{i}$ 's be

$$
A:=\bigcup_{i=0}^{n}\left(A_{i} \times\left\{e_{i}\right\}\right) \subset \mathbb{Z}^{2 n}, e_{i} \in \mathbb{N}^{n}
$$

where $e_{0}, \ldots, e_{n}$ form an affine basis of $\mathbb{R}^{n}$ : $e_{0}$ is the zero vector, $e_{i}=(0, \ldots, 0,1,0, \ldots, 0), i=1, \ldots, n$.
Proposition 3. [Cayley trick] GKZ94] There exist bijections between: the regular tight mixed subdivisions, the tight mixed subdivisions, or the mixed subdivisions of the convex hull of $A_{0}+\cdots+A_{n}$ and, respectively, the regular triangulations, the triangulations, or the polyhedral subdivisions of $A$.

The set of all regular triangulations corresponds to the vertices of the secondary polytope $\Sigma(A)$ of $A$ [GKZ94]. To compute the resultant polytope, one can enumerate all regular triangulations of $A$ : it is equivalent to enumerating all regular tight mixed subdivisions of the convex hull of $A_{0}+\cdots+A_{n}$. Each such subdivision yields a vertex of $N(\mathcal{R})$. This method is proven to be inefficient even for medium sized inputs EK03; instead, we follow a different approach.

### 3.2 The implicit polytope

To predict the Newton polytope of the implicit equation, or implicit polytope, we use the following computation of resultant polytopes and their orthogonal projections, see [EFKP11]. Note that the latter correspond to generic specializations of the resultant.

Given the supports $A_{i}, i=0, \ldots, n$ of the polynomials in (5), there is an efficient algorithm that computes the resultant polytope $N(\mathcal{R})$ of their sparse resultant $\mathcal{R}$ without enumerating all mixed subdivisions of the convex hull of $A_{0}+\cdots+A_{n}$ EFKP11. More precisely, they develop an incremental algorithm to compute $N(\mathcal{R})$ by considering an equivalence relation on mixed subdivisions, where two subdivisions are equivalent iff they specify the same resultant vertex. The class representatives are vertices of the resultant polytope. The algorithm exactly computes vertex- and halfspace-representations of the resultant polytope or its projection. It avoids computing $\Sigma(A)$, but uses the above relationships to define an oracle producing resultant vertices in a given direction. It is output-sensitive as it computes one mixed subdivision per equivalence class, and is the fastest today in dimension up to 5 ; in higher dimensions it is competitive with the implementation of JY11, relying on the GFan library. Moreover, there is an approximate variant that computes polytopes with about $90 \%$ of the true volume with a speedup of up to 25 times; this should be very relevant to approximate implicitization with supports which may be smaller than the exact support.

Let us formalize the way that the polytope $N(\mathcal{R})$ is used in implicitization. Consider an epimorphism of rings

$$
\begin{equation*}
\phi: K \rightarrow K^{\prime}: c_{i j} \mapsto c_{i j}^{\prime}, \tag{6}
\end{equation*}
$$

yielding a generic specialization of the coefficients $c_{i j}$ of the polynomial system in (5). We denote by $F_{i}^{\prime}:=$ $\phi\left(F_{i}\right), i=0, \ldots, n$, the images of $F_{i}$ 's under $\phi$. Let $\mathcal{R}:=\operatorname{Res}\left(F_{0}, \ldots, F_{n}\right)$ be the resultant of polynomial system in (5) over $K$ and $H:=\operatorname{Res}\left(F_{0}^{\prime}, \ldots, F_{n}^{\prime}\right)$ be the resultant of $F_{0}^{\prime}, \ldots, F_{n}^{\prime}$ over $K^{\prime}$. Then, the specialized sparse resultant $\phi(\mathcal{R})$ coincides (up to a scalar multiple from $K^{\prime}$ ) with the resultant $H$ of the system of specialized polynomials provided that $H$ does not vanish, a certain genericity condition is satisfied, and the parametrization is generically 1-1 [LO98, SY94, thm.3]:

$$
\begin{equation*}
\phi(\mathcal{R})=c \cdot H, c \in K^{\prime} . \tag{7}
\end{equation*}
$$

If the latter condition fails, then $\phi(\mathcal{R})$ is a power of $H$. When the genericity condition fails for a specialization of the $c_{i j}$ 's, the support of the specialized resultant $\phi(\mathcal{R})$ is a superset of the support of $H$ modulo a translation, provided the sparse resultant does not vanish. This follows from the fact that the method computes the same polytope as the tropical approach, whereas the latter is characterized in prop. 1. In particular, the resultant polytope is a Minkowski summand of the fiber polytope $\Sigma_{\pi}(\Delta, P)$, where polytope $\Delta$ is a product of simplices, each corresponding to a support $A_{i}, P=\sum_{i=0}^{n} P_{i}$, and $\pi$ is a projection from $\Delta$ onto $P$. Then, $\Sigma(\Delta, P)$, is strongly isomorphic to the secondary polytope of the point set obtained by the Cayley embedding of the $A_{i}$ 's,

Stu94, sec.5]. The algorithm in EFKP11] provides the Newton polytope of $\phi(\mathcal{R})$, and is employed by our approach.

When specialization $\phi$ yields the coefficients of the polynomials in (3), i.e. $\phi\left(F_{i}\right)=\bar{F}_{i}$, then $H=$ $\operatorname{Res}\left(\bar{F}_{0}, \ldots, \bar{F}_{n}\right)=p\left(x_{0}, \ldots, x_{n}\right)$, where $p\left(x_{0}, \ldots, x_{n}\right)$ is the implicit equation of the hypersurface defined by (3). Equation (7) reduces to

$$
\phi(\mathcal{R})=c \cdot p\left(x_{0}, \ldots, x_{n}\right), c \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]
$$

hence EFKP11] yields a superset of the vertices of the implicit polytope. The coefficients of the polynomials in (5), which define the projection $\phi$, are those who are specialized in linear polynomials in the $x_{i}$ 's.

The above discussion is summarized in the following result, which offers the theoretical basis of our approach.
Lemma 4. Given a parametric hypersurface, we formulate implicitization as an elimination problem, thus defining the corresponding sparse resultant. The projection of the sparse resultant's Newton polytope contains a translate of the Newton polytope of the implicit equation.

Let us now give two techniques for improving our approach. The following lemma is used at preprocessing before support prediction, since it reduces the size of the input supports.

Lemma 5. JY11, lem.3.20] If $a_{i j} \in A_{i}$ corresponds to a specialized coefficient of $F_{i}$, and lies in the convex hull of the other points in $A_{i}$ corresponding to specialized coefficients, then removing $a_{i j}$ from $A_{i}$ does not change the Newton polytope of the specialized resultant.

Furthermore, in order to eliminate some extraneous monomials predicted by our support prediction method, we may apply the following well-known degree bounds, generalized in the context of sparse elimination. For a proof, the reader may refer to [EK03].

Proposition 6. The total degree of the implicit polynomial of the hypersurface corresponding to system (3) is bounded by $n$ ! times the volume of the convex hull of $A_{0} \cup \cdots \cup A_{n}$. The degree of the implicit polynomial in some $x_{j}, j \in\{0, \ldots, n\}$ is bounded by the mixed volume of the $\bar{F}_{i}, i \neq j$, seen as polynomials in $t$.

The classical results for the dense case follow as corollaries. Take a surface parameterized by polynomials of degree $d$, then the implicit polynomial is of degree $d^{2}$. For tensor parameterizations of bi-degree $\left(d_{1}, d_{2}\right)$, the implicit degree is $2 d_{1} d_{2}$. We use these bounds to reduce the predicted Newton polytope by intersecting it with the halfspaces prescribed by the proposition.

The resultant polytope $N(\mathcal{R})$ lies in $\mathbb{R}^{|A|}$ but we shall see that it is of lower dimension. Let us describe the hyperplanes in whose intersection lies $N(\mathcal{R})$. For this, let $\mathcal{A}$ be the $(2 n+1) \times|A|$ matrix whose columns are the points in the $A_{i}$, where each $a \in A_{i}$ is followed by the $i$-th unit vector in $\mathbb{N}^{n+1}$.

Proposition 7. GKZ94 $N(\mathcal{R})$ is of dimension $|A|-2 n-1$. The inner product of any coordinate vector of $N(\mathcal{R})$ with row $i$ of $\mathcal{A}$ is: constant, for $i=1, \ldots, n$, and equals the mixed volume of $F_{0}, \ldots, F_{j-1}, F_{j+1}, \ldots, F_{n}$, for $j=i-(n+1), i=n+1, \ldots, 2 n+1$.

The last $n+1$ relations specify the fact that $\mathcal{R}$ is separately homogeneous in the coefficients of each $F_{i}$. The proposition implies that one obtains an isomorphic polytope when projecting $N(\mathcal{R})$ along $2 n+1$ points in $\cup_{i} A_{i}$, which affinely span $\mathbb{R}^{2 n}$; this is possible because of the assumption that $\left\{A_{0}, \ldots, A_{n}\right\}$ is an essential family. Having computed the projection, we obtain $N(\mathcal{R})$ by computing the missing coordinates as the solution of a linear system: we write the aforementioned inner products as $\mathcal{A}[X V]^{T}=C$, where $C$ is a known matrix and $[X V]^{T}$ is a transposed $|A| \times u$ matrix, expressing the partition of the coordinates to unknown and known values, where $u$ is the number of $N(\mathcal{R})$ vertices. If the first $2 n+1$ columns of $\mathcal{A}$ correspond to specialized coefficients, $\mathcal{A}=\left[\mathcal{A}_{1} \mathcal{A}_{2}\right]$, where submatrix $\mathcal{A}_{1}$ is of dimension $2 n+1$ and invertible, hence $X=\mathcal{A}_{1}^{-1}\left(C-\mathcal{A}_{2} V\right)$.

Knowledge of the resultant support can reduce resultant computation to interpolation of coefficients identical to the kind of interpolation developed here for implicitization; this is also the premise of [CD06, Tan07]. To sample points on the resultant hypersurface, one can use a parametrization of the resultant hypersurface, known as Horn-Kapranov parametrization Kap91, illustrated below.

Example 1. Let $f_{0}=a_{2} x^{2}+a_{1} x+a_{0}, f_{1}=b_{1} x^{2}+b_{0}$, with supports $A_{0}=\{2,1,0\}, A_{1}=\{1,0\}$. Their (Sylvester) resultant is a polynomial in $a_{2}, a_{1}, a_{0}, b_{1}, b_{0}$. The algorithm in EFKP11 computes its Newton polytope with vertices $(0,2,0,1,1),(0,0,2,2,0),(2,0,0,0,2)$; it contains 4 points, corresponding to 4 potential monomials $a_{1}^{2} b_{1} b_{0}, a_{0}^{2} b_{1}^{2}, a_{2} a_{0} b_{1} b_{0}, a_{2}^{2} b_{0}^{2}$. The Horn-Kapranov parameterization of the resultant yields: $a_{2}=$ $\left(2 t_{1}+t_{2}\right) t_{3}^{2} t_{4}, a_{1}=\left(-2 t_{1}-2 t_{2}\right) t_{3} t_{4}, a_{0}=t_{2} t_{4}, b_{1}=-t_{1} t_{3}^{2} t_{5}, b_{0}=t_{1} t_{5}$, where the $t_{i}$ 's are parameters. We substitute these expressions to the predicted monomials, evaluate at 4 sufficiently random $t_{i}$ 's, and obtain a matrix whose kernel vector $(1,1,-2,1)$ yields $\mathcal{R}=a_{1}^{2} b_{1} b_{0}+a_{0}^{2} b_{1}^{2}-2 a_{2} a_{0} b_{1} b_{0}+a_{2}^{2} b_{0}^{2}$.

The complexity of interpolating resultants is $O^{*}\left(|S|^{2}\right)$ where $S$ is the set of lattice points in the predicted resultant support, because the dominating stage is a kernel computation for a structured matrix $M$. Using

Weidemann's approach, the main oracle is post-multiplication of $M$ by a vector, which amounts to evaluating a $(n+1)$-variate polynomial at chosen points, and this can be done in quasi-linear complexity in $|S|$ vdHS10, Pan94. For certain classes of polynomial systems, when one computes the resultant in one or more parameters, this may be competitive to current methods for resultant computation. The best such methods rely on developing the determinant of a resultant matrix in these parameters [CE00, D'A02. The matrix dimension is in $O^{*}\left(t^{n} \operatorname{deg} \mathcal{R}\right)$ Emi96, where $\operatorname{deg} \mathcal{R}$ is the total degree of $\mathcal{R}$ in all input coefficients, and $t$ is the scaling factor relating the input Newton polytopes, which is bounded by the maximum degree of the input polynomials $f_{i}$ in any variable. Then, developing univariate resultants has complexity in $O^{*}\left(t^{3.5 n}(\operatorname{deg} \mathcal{R})^{3.5}\right)$ Emi96, EP02, EP05]. Hence, our approach improves the complexity when the predicted support is small compared to $t$ and $\operatorname{deg} \mathcal{R}$, but further work on this topic is required.

## 4 Implicitization algorithm

The steps of our implicitization algorithm are given below, and apply to any support prediction method.
Input: Polynomial or rational parametrization $x_{i}=f_{i}\left(t_{1}, \ldots, t_{n}\right)$.
Output: Implicit polynomial $p\left(x_{i}\right)$ in the monomial basis in $\mathbb{N}^{n+1}$.

1. We obtain (a superset of) the implicit polytope, and intersect it with the halfspaces described in prop. 6
2. Compute all lattice points $S \subseteq \mathbb{N}^{n+1}$ in the polytope.
3. Repeat $\mu \geq|S|$ times: Select value $\tau \in \mathbb{C}^{n}$ for $t$, evaluate $x_{i}(\tau), i=0, \ldots, n$, then evaluate each monomial in $S$.
4. Construct the $\mu \times|S|$ matrix $M$, solve $M \vec{p}=0$ for $p$, and return the primitive part of polynomial $\vec{p}^{\top} S$.

### 4.1 Building the matrix

We focus on two support prediction methods. The first applies only to curves and is described in EKP10. The second is general and computes the support of the resultant of system (3) and of its arbitrary specializations EFKP11]. Both methods provide us a (super)set of the implicit vertices: the set of vertices of the polytope $N(\phi(\mathcal{R}))$ of the specialized resultant $\phi(\mathcal{R})$, where $\mathcal{R}$ is the resultant of the system of polynomials in (5). This polytope is then intersected with the halfspaces described in prop. 6. In the following, we abuse notation and denote this intersection also by $N(\phi(\mathcal{R}))$.

We compute the $m$ lattice points $s_{i}$ contained in $N(\phi(\mathcal{R})) \subset \mathbb{N}^{n+1}$ to obtain the set $S:=\left\{s_{1}, \ldots, s_{m}\right\}$. Each $s_{i}=\left(s_{i 0}, \ldots, s_{i n}\right)$ is an exponent of a (potential) monomial $x^{s_{i}}=x_{0}(t)^{s_{i 0}} \ldots x_{n}(t)^{s_{i n}}$ of the implicit polynomial, where $x_{i}(t)$ is given in (2) . We evaluate $x_{i}(t), i=0, \ldots, n$ at some $\tau_{k}, k=1, \ldots, m$. Let $x\left(\tau_{k}\right)^{s_{i}}=x_{0}\left(\tau_{k}\right)^{s_{i 0}} \ldots x_{n}\left(\tau_{k}\right)^{s_{i n}}$ denote the evaluated $i$-th monomial $x^{s_{i}}$ at $\tau_{k}$. Thus, we construct an $m \times m$ matrix $M$ with rows indexed by $\tau_{1}, \ldots, \tau_{m}$ and columns by $s_{1}, \ldots, s_{m}$ :

$$
M=\left[\begin{array}{ccc}
x\left(\tau_{1}\right)^{s_{1}} & \cdots & x\left(\tau_{1}\right)^{s_{m}} \\
\vdots & \cdots & \vdots \\
x\left(\tau_{m}\right)^{s_{1}} & \cdots & x\left(\tau_{m}\right)^{s_{m}}
\end{array}\right]
$$

To improve the numerical stability of our algorithm, we may use $\mu>m$ evaluation points, so as to arrive at a $\mu \times m$ matrix $M$.

Lemma 8. Assume that we construct a $\mu \times m$ matrix $M$, as above. Then, the vector of the implicit coefficients lies in the matrix kernel, hence $\operatorname{rank}(M)<m$. If the points $x\left(\tau_{i}\right), i=1, \ldots, \mu$ are sufficiently generic, then $M$ has corank 1, i.e. $\operatorname{rank}(M)=m-1$. Then, if we solve $M \vec{p}=\overrightarrow{0}$ for $\vec{p}$, such that any of its entries equals 1, this yields the coefficients of the implicit equation in a unique fashion.

### 4.2 Complexity

In this section we briefly analyze the asymptotic complexity of the main subroutines of our algorithm.
The complexity of the support prediction algorithm is given in [EFKP11, thm.10]. The second part of the procedure is the computation of the lattice points contained in the predicted polytope. It is a NP-hard problem to detect a lattice point in a polytope when the dimension of the polytope is an input variable. When the dimension is fixed the algorithm in BP99 counts the number of lattice points in a polytope within polynomial time in the size of the input. The software LattE $\left[\mathrm{LHH}^{+} 03\right]$ implements Barnivok's algorithm. The software package Normaliz [BIS] computes lattice points in polytopes, and is very fast in practice; this is the one interfaced
to our software. Based on these algorithms, one can enumerate all lattice points in output-sensitive manner, i.e. in polynomial time in the output size, which of course can be exponential in the input size.

Suppose that, for the predicted support $S$, the exponent of every monomial in the $i$-th variable lies in $[0, \delta]$, for $i=1,2, \ldots, n$. Let $O^{*}(\cdot)$ denote asymptotic bounds when ignoring polylogarithmic factors in the arguments.

Proposition 9. EP02, lem.4.3] Consider a set $S$ of monomials in $n$ variables. Given $n$ scalar values $p_{1}, p_{2}, \ldots, p_{n}$, the algorithm of [EP02] evaluates all the monomials of $S$ at these values in $O^{*}(m n+n \sqrt{\delta})$ arithmetic operations and $O(m n)$ space.

Now, we arrive at the complexity of constructing a $\mu \times m$ matrix $M$, with columns indexed by monomials and rows indexed by $\mu$ values.

Corollary 10. Assume our algorithm builds a rectangular matrix $\mu \times m, \mu \geq m$. Then, all $\mu m$ entries are computed in $O^{*}(\mu m n)$ operations.

Once constructed, the kernel computation costs $O\left(m^{2.376}\right)$ arithmetic operations, which follows from the current record for matrix multiplication. Our bound can be improved if matrix multiplication is improved. On $\mu \times m$ rectangular matrices, the kernel computation has complexity $O\left(\mu \mathrm{~m}^{2}\right)$.

An interesting aspect is that $M$ has the structure of quasi-Vandermonde matrices. In particular, multiplying $M$ by a vector $v$ on the right-hand side is equivalent to evaluating a $(n+1)$-variate polynomial with support $S$ and coefficient vector $v$ at all points defining the rows of $M$. If this complexity were quasilinear in $m$ (i.e. linear when ignoring polylogarithmic factors in $m$ ), then the kernel computation of $M$ should have complexity quasi-quadratic in $m$, or even faster, when computing only one eigenvector by Lanczos' method.

### 4.3 Examples

We conclude this section with some examples so as to illustrate our algorithm.
Example 2 (Folium of Descartes). Let us consider the following curve.

$$
x_{0}=\frac{3 t^{2}}{t^{3}+1}, x_{1}=\frac{3 t}{t^{3}+1} .
$$

The algorithm in EKP10 yields 3 implicit polytope vertices: $[1,1],[0,3],[3,0]$. This polygon contains 5 lattice points which yield the potential implicit monomials $x_{1}^{3}, x_{0} x_{1}, x_{0} x_{1}^{2}, x_{0}^{2} x_{1}, x_{0}^{3}$ indexing the columns of matrix $M$ in this order. To fill the rows of matrix $M$, we plug in to each monomial the parametric expressions and evaluate using 5 random integer $\tau$ 's: $19,17,10,6,16$. Then,

$$
M=\left[\begin{array}{ccccc}
\frac{1270238787}{322828856000} & \frac{61731}{47059600} & \frac{66854673}{322828856000} & \frac{3518667}{322828856000} & \frac{185193}{322828856000} \\
\frac{24137569}{4394826072} & \frac{4913}{2683044} & \frac{1419857}{4394826072} & \frac{83521}{4394826072} & \frac{4913}{4394826072} \\
\frac{27000000}{1003003001} & \frac{9000}{1002001} & \frac{2700000}{1003003001} & \frac{270000}{1003003001} & \frac{27000}{1003003001} \\
\frac{1259712}{10218313} & \frac{1944}{47089} & \frac{209952}{10218313} & \frac{34992}{10218313} & \frac{5832}{10218313} \\
\frac{452984832}{68769820673} & \frac{36864}{16785409} & \frac{28311552}{68769820673} & \frac{1769472}{68769820673} & \frac{110592}{68769820673}
\end{array}\right]
$$

The nullvector is $[1,-3,0,0,1]$ : its 3 nonzero entries correspond to monomials $x_{1}^{3}, x_{0} x_{1}, x_{0}^{3}$, i.e. the actual monomials of the implicit equation. The latter turns out to be $x_{0}^{3}-3 x_{0} x_{1}+x_{1}^{3}$, which equals the true implicit equation of the curve.

Example 3 (Bicubic surface). We consider the benchmark challenge of the bicubic surface GV97:

$$
\begin{gathered}
x_{0}=3 t_{1}\left(t_{1}-1\right)^{2}+\left(t_{2}-1\right)^{3}+3 t_{2}, x_{1}=3 t_{2}\left(t_{2}-1\right)^{2}+t_{1}^{3}+3 t_{1} \\
x_{2}=-3 t_{2}\left(t_{2}^{2}-5 t_{2}+5\right) t_{1}^{3}-3\left(t_{2}^{3}+6 t_{2}^{2}-9 t_{2}+1\right) t_{1}^{2}+t_{1}\left(6 t_{2}^{3}+9 t_{2}^{2}-18 t_{2}+3\right)-3 t_{2}\left(t_{2}-1\right)
\end{gathered}
$$

The implicit degree in $x_{0}, x_{1}$ is 18 , and 9 in $x_{2}$. The approach of EK03 could not handle it because it generates 737129 regular triangulations (by TOPCOM) in a file of 383 MB ; our method computes the optimal support. The implicit polytope has vertices $[0,0,0],[18,0,0],[0,18,0],[0,0,9]$, and 715 lattice points. The nullvector of matrix $M$ contains 715 non-zero entries, which correspond precisely to the actual implicit support. It is computed in 47 sec .


Figure 2: Folium of Descartes.

Hypercone. We illustrate our method on a hypersurface of dimension 4, whose rational representation is:

$$
\begin{equation*}
x_{0}=\frac{r\left(1-t^{2}\right)\left(1-s^{2}\right)}{\left(1+t^{2}\right)\left(1+s^{2}\right)}, x_{1}=\frac{2 r\left(1-t^{2}\right) s}{\left(1+t^{2}\right)\left(1+s^{2}\right)}, x_{2}=\frac{2 r t}{1+t^{2}}, x_{3}=r \tag{8}
\end{equation*}
$$

This is an example where the resultant of the system (3) is a multiple of the implicit equation, hence it defines a variety strictly containing the image of the parametrization. The extraneous factor is the variety defined by the denominators of polynomials in (8). Adding the polynomial in equation (4) yields the exact variety. In order to facilitate the computation of the predicted support, we set up an equivalent system:

$$
\begin{array}{r}
F_{0}=x_{0} w-r\left(1-t^{2}\right)\left(1-s^{2}\right), F_{1}=x_{1} w-2 r\left(1-t^{2}\right) s, \\
F_{2}=x_{2} w-2 r t\left(1+s^{2}\right), F_{3}=x_{3}-r  \tag{9}\\
F_{4}=w-\left(1+s^{2}\right)\left(1+t^{2}\right), F_{5}=1-w y
\end{array}
$$

where $w$ is a new variable. We then compute the support of the system's sparse resultant by eliminating $r, s, t, w$, projected to the space of $x_{0}, x_{1}, x_{2}, x_{3}$. The software from EFKP11 predicts 4 implicit vertices: $[8,0,0,0],[0,8,0,0],[0,0,8,0],[0,0,0,8]$. This polytope contains 165 lattice points, all of which correspond to actual monomials of a polynomial of total degree 8 , which is multiple of the true the equation of the hypercone. This equation equals $x_{0}^{2}+x_{1}^{2}+x_{2}^{2}-x_{3}^{2}$. This is an extreme example showing how our method may produce a multiple of the true implicit equation.

## 5 Implementation and experimental results

This section looks at the actual symbolic and numeric computations once the problem has been reduced to a question in linear algebra. We start with software for the matrix operations, then discuss different ways to evaluate the matrix entries, how to measure the accuracy of approximate implicitization, and compare with other approaches. Our algorithms are implemented in Maple and Matlab.

Let us refer to lemma 8 and assume $M$ has corank 1. Solving the linear system

$$
M \vec{p}=\overrightarrow{0}
$$

yields the implicit coefficient $p_{i}$ for each predicted monomial $x^{s_{i}}$. The kernel null( $M$ ) is 1-dimensional, hence some entry $p_{i}$ is set to 1 . We form the inner product of the vector of the monomials indexing the columns of $M$ with $\vec{p}$, and then take the primitive part of the resulting polynomial to define the implicit equation. Of course, such exact methods can treat indefinite parameters which may be encountered in parametric expressions.

For larger examples, we trade exactness for speed and apply Singular Value Decomposition (SVD), thus computing

$$
M \vec{p}^{\top}=\left(U \Sigma V^{\top}\right) \vec{p}^{\top}=\overrightarrow{0}^{\top} \Leftrightarrow \Sigma \vec{v}^{\top}=\overrightarrow{0}^{\top}, \text { where } V \vec{v}^{\top}=\vec{p}^{\top},
$$

where $U U^{\top}=V V^{\top}=I$ and $\Sigma$ is diagonal. A basis of $\operatorname{null}(M)$ consists of the last columns of $V$ corresponding to the zero singular values of $M$, because $V$ is orthogonal. When $\operatorname{corank}(M)=1, v=[0, \ldots, 0,1]$ and the last row of $V^{\top}$ gives $\vec{p}$. The same derivation holds if $M$ is rectangular, say $\mu \times m, \mu \geq m$. Then $\Sigma$ is of the same dimensions, $U$ is $\mu \times \mu$, and $V$ is $m \times m$, where its last column is the sought vector.

Our algorithm is implemented in Maple 13. For exact kernel computation, we use function LinearSolve() from package LinearAlgebra, or function Linear() from package SolveTools. Equivalently, we may compute null( $M$ ) using the command NullSpace() of LinearAlgebra. SVD is implemented with command SingularValues().

We have also implemented numerical versions of our algorithm in Matlab. The numerical stability of matrix $M$ is measured by comparing ratios of singular values of $M$. We employ the condition number $\kappa(M)=\sigma_{1} / \sigma_{m}$, as well as ratio $\sigma_{1} / \sigma_{m-1}$, where $\sigma_{1}$ is the maximum singular value. By comparing these two numbers, we decide whether the matrix is of numerical corank 1 , otherwise we instantiate a new matrix using new values.

All experiments, unless otherwise stated, were performed on a Celeron 1.6 GHz linux machine with 1 GB of memory. Most curves and surfaces in our experiments are in Table 7, and Table 8 in the Appendix. The tables show the parametric, the exact implicit and the approximate implicit representation. The last two equations are primitive and for the latter we omit terms with very small coefficients. Runtimes (sec) for the various approaches to implicitization of these curves and surfaces are given, respectively, in Table 1, and Table 2, on Maple. In both tables, we used random integers for exact computation, with functions NullSpace and LinearSolve, and unitary complexes for numeric computation with SVD.

We also used dense and sparse Bézier curves of various degrees; the runtimes for this family of curves are shown in Table 4. In this set of experiments we show the size $\mu \times m, \mu>m$ of matrices used in numerical computation; the corresponding matrices for exact computation are $m \times m$.

A first observation is that SVD and exact linear algebra are competitive on our inputs; we expect that SVD shall be significantly faster on larger inputs. A second observation is that the approximate methods we applied gave very satisfactory results with respect to the accuracy of the computed implicit equation. Notice that in certain cases we use a rectangular $\mu \times m, \mu>m$ matrix so as to improve numerical stability. Overall, our results are encouraging and indicate that the algorithms in this paper are worth applying to implicitization. However, as the matrix size grows, our current implementations show their limitations.

| Curve | Exact |  |  | SVD |  |  | \#impl. |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | NullSpace | LinearSolve | matrix size | time | accuracy (a) | matrix size | monom. |

Table 1: Runtimes (sec) and accuracy of approximation for curves.

| Surface | Exact |  | SVD |  | matrix | \# implicit <br> monomials |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | NullSpace | LinearSolve | time | accuracy (a) | size |  |

Table 2: Runtimes (sec) and accuracy of approximation for surfaces.

### 5.1 Point sampling

A central part in our linear system construction is held by the evaluation of matrix $M$ at convenient $\tau$. This section describes our approaches and experimental results.

We have experimented with both integer and complex values. In the former case, we used random and mutually prime integers to achieve exactness. The chosen value is discarded if it makes some denominator vanish among the parametric expressions. We also tried complex values for $\tau$ : Given an $m \times m$ matrix, we used $2 m$-th roots of unity, and random unitary complexes, i.e. complex numbers of modulus equal to 1 . The roots of unity when used with approximate methods where evaluated as floats. When examining approximate methods we used the ratio of the last two singular values $\sigma_{m} / \sigma_{m-1}$, which indicates how close to having corank 1 is matrix $M$.

Table 3 shows representative timings about these options, which we examined with our implementation on Maple, optimized for the specific task. The experiments were performed on a Intel I3-380UM 1.33 GHz linux
machine with 4 GB of memory. The specifics in the setup of these experiments account for the differences of the corresponding timings shown in other tables.

Our experiments show that runtimes do not vary significantly in small examples but in larger ones, the best results are given by random integers for the exact method, and unitary complexes and roots of unity evaluated as floats for the numeric method, with the former having a slightly better overall performance over the latter, both in terms of stability and speed.

As expected, random integers give matrices which are closer to having corank 1. Note that in Table 3. the Trifolium's matrix $M$, when computed using random integers, was of corank $>1$ but we were still able to compute a multiple of its implicit equation using exact methods. Namely, any kernel vector supplied a multiple of the implicit equation. The degree of the extraneous factor varied depending on the vector chosen.

For the family of Bézier curves, using random integers we obtain better values of the ratio of singular values, compared to other evaluation methods.

| Curve | implicit degree | lattice points | $\operatorname{SVD}\left(\sigma_{m} / \sigma_{m-1}\right)$ |  |  | NullSpace rand.Z |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | root of 1 | unitary $\mathbb{C}$ | rand.Z |  |
| Folium of Descartes | 3 | 5 | $0.136\left(10^{-10}\right)$ | $0.044\left(10^{-6}\right)$ | $0.032\left(10^{-8}\right)$ | 0.032 |
| Conchoid | 4 | 10 | $0.372\left(10^{-12}\right)$ | $0.128\left(10^{-8}\right)$ | $0.096\left(10^{-2}\right)$ | 0.06 |
| Bean curve | 4 | 13 | $0.328\left(10^{-6}\right)$ | $0.092\left(10^{-2}\right)$ | 0.06 ( $10^{-3}$ ) | 0.108 |
| Trichoid | 4 | 15 | $0.5\left(10^{-8}\right)$ | $0.296\left(10^{-3}\right)$ | $0.472\left(10^{-1}\right)$ | 0.116 |
| Cardioid | 4 | 15 | $0.764\left(10^{-6}\right)$ | $0.364\left(10^{-5}\right)$ | $0.184\left(10^{-1}\right)$ | 0.132 |
| Nephroid | 6 | 28 | $1.844\left(10^{-5}\right)$ | $1.460\left(10^{-1}\right)$ | $0.144\left(10^{-4}\right)$ | 1.932 |
| Talbot's curve | 6 | 28 | 1.596 ( $10^{-7}$ ) | $1.424\left(10^{-5}\right)$ | $0.124\left(10^{-2}\right)$ | 2.128 |
| Trifolium | 8 | 45 | 1.753 (10 $0^{-40}$ ) | $0.712\left(10^{-8}\right)$ | - | 13.71,corank $>1$ |
| Ranunculoid | 12 | 91 | $59.764\left(10^{-84}\right)$ | $62.424\left(10^{-58}\right)$ | $79.853\left(10^{-2}\right)$ | 8643.816 |
| Dense Bézier deg. 8 | 8 | 45 | $1.884\left(10^{-2}\right)$ | $0.848\left(10^{-1}\right)$ | $7.329\left(10^{-225}\right)$ | 18.597 |
| Sparse Bézier deg. 8 | 8 | 33 | $1.072\left(10^{-2}\right)$ | $0.404\left(10^{-5}\right)$ | $0.148\left(10^{-145}\right)$ | 2.984 |

Table 3: Comparison of matrix evaluation methods. Runtimes on Maple (sec), whereas the parenthesis contains $\sigma_{m} / \sigma_{m-1}$.

### 5.2 Accuracy of approximate implicitization

In this section, we evaluate the numeric accuracy, or quality, of the approximate implicit equation obtained by our method, by comparing it to the exact implicit equation.

When using numerical methods, the computed implicit equation is not a polynomial with rational coefficients, hence we need to convert the computed real or complex kernel-vector to a rational vector. This is achieved by setting all coefficients smaller than a certain threshold, defined by the problem's condition number, equal to zero. The result is not always equal to the exact implicit equation, so its accuracy is quantified by two measures discussed later. The overall process is computationally rather costly; it can be avoided whenever an implicit equation with floating point coefficients is sufficient for a specific application.

We employ two measures to quantify the accuracy of approximate implicitization:
(a) Coefficient difference: measured as the norm of the difference of the two coefficient vectors $V_{\text {exact }}, V_{\text {app }}$, obtained from exact and approximate implicitization, after padding with zero the entries of each vector which do not appear in the other.
(b) Evaluation norm: measured by considering the maximum norm of the approximate implicit equation when evaluated at a set of sampled points on the given parametric object. This is of course a lower bound on how far from zero can such a value be.

We can actually improve the accuracy of approximation if we disregard all real or complex entries of the coefficient vector with norm close to zero. This simple filtering, applied with a threshold of $10^{-6}$, improves the accuracy under measure (a) by up to one order of magnitude. All results shown in the tables concerning approximate implicitization make use of this filtering.

The approximate implicit equation in all experiments below is obtained using the command SingularValues(), where the matrix is instantiated by unitary complex values $\tau$, whereas the exact one is obtained using command NullSpace() using random integers. We used several parametric curves and surfaces. The computed approximate implicit equations are given in Table 7, and respectively Table 8. The runtimes of approximate and exact methods, and the accuracy of approximation using measure (a) above, are shown in Table 1 and Table 2. These results confirm that SVD can give very good approximations of the actual implicit equation in most inputs,
despite the fact that the current implementation has difficulties dealing with large inputs. It is the direction we are following in the future, combined with further algorithmic improvements such as a smaller support.

One of the main difficulties of approximating the implicit equation is to build the matrix $M$ so that its numeric corank is 1 . Our experiments indicate, expectedly, that if the entries of $M$ take big absolute values, then computations with $M$ are less stable. We improve stability by avoiding values that make the denominators of the parametric polynomials evaluate close to 0 . These values are singular points so we choose a box containing each such point and remove them when we pick different values. Moreover, we add more rows to $M$.

We present some specific examples, using both dense and sparse Bézier curves of varying degree (see example 4), yielding dense and sparse implicit equations. These are polynomial parameterizations, where the implicit equation is of the same total degree. We compare the runtimes for exact and approximate methods, and the accuracy of the latter using both measures: (a) in Table 4. and (b) in Table 5. Both measures give overall very encouraging results.

Table 4 also juxtaposes the efficiency of our algorithm on dense and sparse Bézier inputs. It appears that we are able to exploit sparseness, since the matrix size is smaller in sparse inputs, and not very far from the actual size of the implicit support. This translates into faster runtimes and better accuracy of the approximate implicit equation. For the dense curve of degree 8, the accuracy of the approximate polynomial is rather large, but may be acceptable given the large norm of the coefficient, namely at least $10^{6}$. Figure A in the Appendix summarizes the accuracy estimation, using criterion (a), for approximating dense Bézier curves. The figure shows the hardness of approximation as the parametric and implicit degree grows.

| Implicit <br> degree | SVD |  | NullSpace |  | Accuracy (a) |  | Matrix size (SVD) |  | \# nonzero terms |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | dense | sparse | dense | sparse | dense | sparse | dense | sparse | dense | sparse |
| 4 | 0.696 | 0.244 | 0.148 | 0.060 | $4.874 \cdot 10^{-12}$ | $1.198 \cdot 10^{-17}$ | $30 \times 15$ | $22 \times 11$ | 15 | 8 |
| 5 | 1.616 | 0.584 | 0.740 | 0.228 | $4.184 \cdot 10^{-7}$ | $2.579 \cdot 10^{-22}$ | $42 \times 21$ | $32 \times 16$ | 21 | 14 |
| 6 | 3.788 | 0.464 | 4.768 | 0.908 | $4.843 \cdot 10^{-5}$ | $3.006 \cdot 10^{-11}$ | $56 \times 28$ | $38 \times 19$ | 28 | 19 |
| 8 | 19.173 | 5.704 | 133.148 | 17.141 | 23.717 | $4.175 \cdot 10^{-4}$ | $90 \times 45$ | $66 \times 33$ | 45 | 32 |

Table 4: Maple runtimes ( sec ) and accuracy for dense and sparse Bézier curves.

| Surface | Max norm of approximate implicit polynomial |
| :--- | :---: |
| Bohemian dome | $7.21668 \cdot 10^{-10}$ |
| Quartoid | $7.44845 \cdot 10^{-16}$ |
| Sine | $1.25549 \cdot 10^{-5}$ |
| Swallowtail | $1.98798 \cdot 10^{-10}$ |

Table 5: Accuracy of approximation under measure (b) over 150 sampled points

Example 4. We consider a family of dense and sparse Bézier polynomial curves. Their implicit degree equals the maximum degree of their parametric polynomials. The dense Bézier curve of degree 8 has both parametric polynomials of maximum degree:
$x(t)=4 t-42 t^{2}+168 t^{3}-385 t^{4}+532 t^{5}-406 t^{6}+140 t^{7}-11 t^{8}, y(t)=1 / 2-28 t^{3}+105 t^{4}-196 t^{5}+210 t^{6}-120 t^{7}+29 t^{8}$.
The sparse Bézier curve of degree 8 is missing certain terms, namely one of the parametric polynomials is of lower degree:

$$
x(t)=1+112 t^{3}-630 t^{4}+1344 t^{5}-1344 t^{6}+592 t^{7}-75 t^{8}, y(t)=2-16 t+280 t^{3}-420 t^{4}-392 t^{5}+546 t^{6} .
$$

Further curves of this family are used in our experiments and are generated in a similar fashion. The corresponding accuracy of approximation and the runtimes are shown in Table 4 . Figure A in the Appendix summarizes the approximation accuracy for the dense family.

### 5.3 Comparison to other methods

We report on a preliminary comparison of our implementation against methods using $\mu$-bases [CSC98, BB10, implemented only for curves BB10, and Maple function Implicitize() based on CGKW00. The latter relies on integration of matrix $M$ over each parameter, see sec. 2.1.

Table 6 summarizes the total time to implicitize a curve, given its parameterization. We used the same algebraic curves as in other tables, grouped by degree; for each degree, the table shows the average runtime. In
our experiments, $\mu$-bases yield the fastest runtimes, whereas Implicitize() is the slowest of the three when run in exact mode or when the parametrization is rational.

However, $\mu$-bases rely on exact computation over rational numbers, and an approximate computation would not offer good accuracy. Our algorithm removes this limitation and offers high-quality approximations.

| curve <br> degree | Implicitize <br> exact | Implicitize <br> numeric | Our <br> software | $\mu$-bases |
| :--- | :--- | :--- | :--- | :--- |
| 3 | 2.0465 | 0.031 | 0.024 | 0.016 |
| 4 | 5.386 | 0.056 | 0.035 | 0.031 |
| 6 | 24.854 | 0.117 | 0.109 | 0.036 |
| 8 | 905.68 | 0.249 | 0.266 | 0.047 |
| 12 | $>3000$ | 0.7485 | 1.453 | 0.125 |

Table 6: Comparing runtimes (sec) of: Maple function Implicitize (exact and numeric), our method, and $\mu$-bases.

## 6 Further work

This paper illustrates the merits of our algorithms, and reveals the issues arising in our procedures, using medium-size examples. Some points of current work have been mentioned throughout the paper. Below we focus on certain further ideas for improvement.

In order to tackle large problems we plan to employ state-of-the-art software libraries for matrix operations. For exact computations, LinBox $\mathrm{DGG}^{+} 02$ implements asymptotically optimal algorithms, and is designed to work typically for $\operatorname{dim} M>100$ so as to perform multiple-precision exact computation. It can also be parameterized with single-precision or non-exact number types to yield faster algorithms. Eigen [GJ ${ }^{+}$10] focuses on medium dimensions and in single or multiple precision floating-point computations, and uses modern generic programming techniques to perform optimizations both in compilation and execution time. For approximate implicitization, we plan to use LAPACK for further examination of numerical stability. In this respect, we may consider specific challenges, e.g. CTY10. The authors compute the implicit polytope thus reducing implicitization of a 16-dimensional hypersurface to linear algebra.

We have restricted attention to hypersurfaces, but the algorithms discussed in this paper apply to surfaces of codimension $\geq 2$, such as space curves. In this case, the generalization of the resultant is the Chow form, and our methods could interpolate this form, thus offering information about the implicit representation of the surface. We may also extend our approach to interpolating the implicit polynomial in other bases, such as Bernstein or Lagrange, by predicting the resultant support in these bases.

It is possible to approximate manifolds given by $k$ parametric pieces with a single implicit equation, by applying SVD on $M^{\top}=\left[M_{1} \cdots M_{k}\right]^{\top}$, where $M_{i}$ is the matrix constructed by the algorithms of this paper for the $i$-th piece, for $i=1, \ldots, k$. This includes planar curve or surface splines defined by $k$ segments or patches, respectively. We assume the $k$ parametric representations yield implicit polynomials with (roughly) the same Newton polytope, which always happens if the parametric representation of each piece uses polynomials with the same supports. Matrix $M$ is then evaluated over points spanning all $k$ segments or patches.

Acknowledgements I.Z. Emiris, T. Kalinka, and T. Luu Ba are partially supported by Marie-Curie Initial Training Network "SAGA" (ShApes, Geometry, Algebra), FP7-PEOPLE contract PITN-GA-2008-214584. C. Konaxis enjoys support from the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation".

## References

[APJ11] M. Aigner, A. Poteaux, and B. Juttler. Approximate implicitization of space curves. In U. Langer and P. Paule, editors, Symbolic and Numeric Computation. Springer, Vienna, 2011. To appear.
[BB10] L. Busé and T. Luu Ba. Matrix-based implicit representations of algebraic curves and applications. Computer Aided Geometric Design, 27(9):681-699, 2010.
[BD10a] O.J.D. Barrowclough and T. Dokken. Approximate implicitization and approximate null spaces. The 16th Conference of the International Linear Algebra Society (ILAS), Pisa, Italy, 2010.
[BD10b] O.J.D. Barrowclough and T. Dokken. Approximate implicitization of triangular Bézier surfaces. In Proceedings of the 26th Spring Conference on Computer Graphics, SCCG '10, pages 133-140, New York, NY, USA, 2010.
[BIS] W. Bruns, B. Ichim, and C. Söger. Normaliz. algorithms for rational cones and affine monoids. Available from http://www.math.uos.de/normaliz.
[BOT88] M. Ben-Or and P. Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In Proc. ACM Symp. Theory of Computing, pages 301-309. ACM Press, New York, 1988.
[BP99] A. Barvinok and J. Pommersheim. An algorithmic theory of lattice points in polyhedra. Complexity, 38:91147, 1999.
[CD06] M.A. Cueto and A. Dickenstein. Some results on inhomogeneous discriminants, 2006. arXiv:math/0610031v2 [math.AG].
[CE00] J.F. Canny and I.Z. Emiris. A subdivision-based algorithm for the sparse resultant. J. ACM, 47(3):417-451, May 2000.
[CGKW00] R.M. Corless, M. Giesbrecht, Ilias S. Kotsireas, and S.M. Watt. Numerical implicitization of parametric hypersurfaces with linear algebra. In Proc. AISC, pages 174-183, 2000.
[CKL89] J.F. Canny, E. Kaltofen, and Y. Lakshman. Solving systems of non-linear polynomial equations faster. In Proc. ACM Intern. Symp. on Symbolic $\mathcal{E B}^{\prime}$ Algebraic Comput., pages 121-128, 1989.
[CLO98] D.A. Cox, J.B. Little, and D. O'Shea. Using Algebraic Geometry, volume 185 of Graduate Texts in Mathematics. Springer-Verlag, NY, 1998.
[CSC98] D.A. Cox, T.W. Sederberg, and F. Chen. The moving line ideal basis of planar rational curves. Comput. Aided Geom. Design, 15 (8):803-827, 1998.
[CTY10] M.A. Cueto, E.A. Tobis, and J. Yu. An implicitization challenge for binary factor analysis. J. Symbolic Computation, 45(12):1296-1315, 2010.
[Cue10] M.A. Cueto. Tropical Implicitization. PhD thesis, Dept Mathematics, UC Berkeley, 2010.
[D'A02] C. D'Andrea. Macaulay-style formulas for the sparse resultant. Trans. of the AMS, 354:2595-2629, 2002.
[DFS07] A. Dickenstein, E.M. Feichtner, and B. Sturmfels. Tropical discriminants. J. AMS, pages 1111-1133, 2007.
$\left[\mathrm{DGG}^{+} 02\right]$ J.-G. Dumas, T. Gautier, M. Giesbrecht, P. Giorgi, B. Hovinen, E. Kaltofen, B. D. Saunders, W. J. Turner, and G. Villard. Linbox: A generic library for exact linear algebra. In Proc. 1st Internat. Congress Math. Software (ICMS), pages 40-50, Beijing, China, 2002.
[DS10] C. D'Andrea and M. Sombra. The Newton polygon of a rational plane curve. Math. in Computer Science, 4(1):3-24, 2010.
[DT03] T. Dokken and J.B. Thomassen. Overview of approximate implicitization. Topics in algebraic geometry and geometric modeling, 334:169-184, 2003.
[EFKP11] I.Z. Emiris, V. Fisikopoulos, C. Konaxis, and L. Peñaranda. An output-sensitive algorithm for computing projections of resultant polytopes. arXiv:1108.5985v2 [cs.SC], 2011.
[EK03] I.Z. Emiris and I.S. Kotsireas. Implicit polynomial support optimized for sparseness. In Proc. Intern. Conf. Computational science appl.: Part III, pages 397-406, Berlin, 2003. Springer.
[EK06] A. Esterov and A. Khovanskiǐ. Elimination theory and newton polytopes. arXiv:0611107[math], 2006.
[EKK11] I.Z. Emiris, T. Kalinka, and C. Konaxis. Implicitization of curves and surfaces using predicted support. In Proc. Inter. Works. Symbolic-Numeric Computation, San Jose, Calif., 2011.
[EKP10] I.Z. Emiris, C. Konaxis, and L. Palios. Computing the Newton polygon of the implicit equation. Mathematics in Computer Science, Special Issue on Computational Geometry and Computer-Aided Design, 4(1):25-44, 2010.
[Emi96] I.Z. Emiris. On the complexity of sparse elimination. J. Complexity, 12:134-166, 1996.
[EP02] I.Z. Emiris and V.Y. Pan. Symbolic and numeric methods for exploiting structure in constructing resultant matrices. J. Symbolic Computation, 33:393-413, 2002.
[EP05] I.Z. Emiris and V.Y. Pan. Improved algorithms for computing determinants and resultants. J. Complexity, Special Issue, 21:43-71, 2005. Special Issue on FOCM-02.
$\left[G J^{+} 10\right]$ G. Guennebaud, B. Jacob, et al. Eigen v3. http://eigen.tuxfamily.org, 2010.
[GKZ94] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Discriminants, Resultants and Multidimensional Determinants. Birkhäuser, Boston, 1994.
[GV97] L. Gonzalez-Vega. Implicitization of parametric curves and surfaces by using multidimensional Newton formulae. J. Symbolic Comput., 23(2-3):137-151, 1997. Parametric algebraic curves and applications (Albuquerque, NM, 1995).
[Jen10] A.N. Jensen. Gfan, a software system for Gröbner fans and tropical varieties, 2010. http://www.math.tu-berlin.de/ jensen/software/gfan/gfan.html.
[JY11] A. Jensen and J. Yu. Computing tropical resultants. arXiv:1109.2368v1[math.AG], 2011.
[Kap91] M. M. Kapranov. A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map. Mathematische Annalen, 290:277-285, 1991.
[KL89] E. Kaltofen and Y. Lakshman. Improved sparse multivariate polynomial interpolation algorithms. In P. Gianni, editor, Proc. ACM Intern. Symp. on Symbolic \& $\mathcal{F}$ Algebraic Comput. 1988, volume 358 of Lect. Notes in Comp. Science, pages 467-474. Springer-Verlag, 1989.
[KL03] I.S. Kotsireas and E.S.C. Lau. Implicitization of polynomial curves. In Proc. ASCM, pages 217-226, Beijing, 2003.
$\left[L^{+} H^{+} 03\right]$ J.A. De Loera, D. Haws, R. Hemmecke, P. Huggins, J. Tauzer, and R. Yoshida. A user's guide for latte v1.1. Software package LattE is available at http://www.math.ucdavis.edu/ latte/, 2003.
[MM02] A. Marco and J.J. Martinez. Implicitization of rational surfaces by means of polynomial interpolation. CAGD, 19:327-344, 2002.
[Pan94] V.Y. Pan. Simple multivariate polynomial multiplication. J. Symb. Comp., 18:183-186, 1994.
[Ram02] J. Rambau. TOPCOM: Triangulations of point configurations and oriented matroids. In Arjeh M. Cohen, Xiao-Shan Gao, and Nobuki Takayama, editors, Intern. Conf. Math. Software, pages 330340. World Scientific, 2002.
[Sau04] T. Sauer. Lagrange interpolation on subgrids of tensor product grids. Math. of Comput., 73:181-190, January 2004.
[SJ08] M. Shalaby and B. Jüttler. Approximate implicitization of space curves and of surfaces of revolution. In R. Piene and B. Jüttler, editors, Algebraic Geometry and Geometric Modeling, pages 215-228. Springer, 2008.
[Stu94] B. Sturmfels. On the Newton polytope of the resultant. J. Algebraic Combin., 3:207-236, 1994.
[STY07] B. Sturmfels, J. Tevelev, and J. Yu. The Newton polytope of the implicit equation. Moscow Math. J., 7(2), 2007.
[SY94] B. Sturmfels and J.T. Yu. Minimal polynomials and sparse resultants. In F. Orecchia and L. Chiantini, editors, Proc. Zero-dimensional schemes (Ravello, 1992), pages 317-324. De Gruyter, 1994.
[SY08] B. Sturmfels and J. Yu. Tropical implicitization and mixed fiber polytopes. In Software for Algebraic Geometry, volume 148 of IMA Volumes in Math. $\mathcal{E}$ its Applic., pages 111-131. Springer, New York, 2008.
[Tan07] S. Tanabe. On Horn-Kapranov uniformisation of the discriminantal loci. Adv. Studies Pure Math., 46:223-249, 2007.
[vdHS10] J. van der Hoeven and E. Schost. Multi-point evaluation in higher dimensions. Technical Report 00477658, HAL, 2010.
[WTJD04] E. Wurm, J.B. Thomassen, B. Juttler, and T. Dokken. Comparative benchmarking of methods for approximate implicitization. In M. Neamtu and M. Lucian, editors, Geometric Modeling and Computing, Seattle 2003, pages 537-548. Nashboro Press, 2004.
[Zip90] R. Zippel. Interpolating polynomials from their values. J. Symbolic Computation, 9:375-403, 1990.
[Zip93] R. Zippel. Effective Polynomial Computation. Kluwer Academic Publishers, Boston, 1993.

## A Appendix

| Curve | Parametric form | Exact implicit polynomial | Approximate implicit polynomial |
| :---: | :---: | :---: | :---: |
| Nephroid | $\begin{aligned} & \frac{-\left(-1+t^{2}\right)\left(1+10 t^{2}+t^{4}\right)}{\left(1+t^{2}\right)^{3}} \\ & \frac{32 t^{3}}{\left(1+t^{2}\right)^{3}} \end{aligned}$ | $\begin{aligned} & \hline \hline-64-60 y^{2}-12 y^{4}+y^{6} \\ & +48 x^{2}-24 x^{2} y^{2}+3 x^{2} y^{4}-12 x^{4} \\ & +3 x^{4} y^{2}+x^{6} \end{aligned}$ | $\begin{aligned} & \hline-1.81402 .10^{-5} x^{2} y-23.99999 x^{2} y^{2} \\ & -11.99999 x^{4}-59.99999 y^{2} \\ & -2.17512 .10^{-5} y^{3}-11.99999 y^{4} \\ & +47.99999 x^{2}+3 x^{4} y^{2}+y^{6} \\ & +x^{6}-63.99999+3 x^{2} y^{4} \\ & \hline \end{aligned}$ |
| Talbot's curve | $\begin{aligned} & \frac{-\left(1+6 t^{2}+t^{4}\right)\left(-1+t^{2}\right)}{\left(1+t^{2}\right)^{3}} \\ & \frac{-2 t\left(1-2 t^{2}+t^{4}\right)}{\left(1+t^{2}\right)^{3}} \end{aligned}$ | $\begin{aligned} & x^{6}+3 y^{2} x^{4}-x^{4}+3 y^{4} x^{2} \\ & -20 y^{2} x^{2}+y^{6}+8 y^{4}+16 y^{2} \end{aligned}$ | $\begin{aligned} & 0.00318649 y+0.0019125 x^{2} y^{3} \\ & +2.999989 x^{2} y^{4}+0.00010255 x^{4} y \\ & +2.999994 x^{4} y^{2}-0.004140395 x^{2} y \\ & -20.00909 x^{2} y^{2}+15.9979 y^{2} \\ & +0.0008008 y^{3}+7.999837 y^{4} \\ & +0.00079661 x^{2}+0.9999945 y^{6}+x^{6} \\ & -1.0001 x^{4}+0.8906879 .10^{-3} y^{5} \end{aligned}$ |
| Tricuspoid | $\frac{-t^{4}-6 t^{2}+3}{\left(1+t^{2}\right)^{2}}, \frac{8 t^{3}}{\left(1+t^{2}\right)^{2}}$ | $\begin{aligned} & -27+18 y^{2}+y^{4}+24 x y^{2}+ \\ & +18 x^{2}+2 x^{2} y^{2}-8 x^{3}+x^{4} \end{aligned}$ | $\begin{aligned} & -27+17.99999 y^{2}+0.99999 y^{4} \\ & +23.99999 x y^{2}+18 x^{2}+x^{4} \\ & +1.99999 x^{2} y^{2}-8 x^{3} \end{aligned}$ |
| Ranunculoid | $\begin{aligned} & -\frac{1-1092 t^{6}+423 t^{8}-54 t^{10}}{\left(1+t^{2}\right)^{6}} \\ & +\frac{13 t^{12}-102 t^{2}+363 t^{4}}{\left(1+t^{2}\right)^{6}} \\ & \frac{8 t^{3}\left(-29+108 t^{2}-78 t^{4}+44 t^{6}+3 t^{8}\right)}{\left(1+t^{2}\right)^{6}} \end{aligned}$ | $\begin{aligned} & -52521875-1286250 x^{2}-1286250 y^{2} \\ & -32025\left(x^{2}+y^{2}\right)^{2}+93312 x^{5} \\ & -933120 x^{3} y^{2}+466560 x y^{4} \\ & -812\left(x^{2}+y^{2}\right)^{3}-21\left(x^{2}+y^{2}\right)^{4} \\ & -42\left(x^{2}+y^{2}\right)^{5}+\left(x^{2}+y^{2}\right)^{6} \\ & \hline \end{aligned}$ |  |

Table 7: Parametric, implicit, and approximate implicit representation of curves; for the latter, we do not show coefficients of absolute value $<10^{-6}$.


Figure 3: Accuracy of implicitization of dense Bézier curves

| Surface | Parametric form | Exact implicit polynomial | Approximate implicit polynomial |
| :---: | :---: | :---: | :---: |
| Quartoid | $t, s,-\left(t^{2}+s^{2}\right)^{2}$ | $z+x^{4}+2 x^{2} y^{2}+y^{4}$ | $z+x^{4}+2 x^{2} y^{2}+y^{4}$ |
| Sine surface | $\begin{aligned} & \frac{2 t}{1+t^{2}}, \\ & \frac{2 s}{1+s^{2}} \\ & \frac{2 s+2 t-2 s t^{2}-2 t s^{2}}{1+s^{2}+t^{2}+s^{2} t^{2}} \end{aligned}$ | $\begin{aligned} & -2 y^{2} z^{2}+4 x^{2} y^{2} z^{2}-2 x^{2} y^{2} \\ & -2 x^{2} z^{2}+z^{4}+y^{4}+x^{4} \end{aligned}$ | $\begin{aligned} & -0.23681 \cdot 10^{-5} y+0.20275 \cdot 10^{-5} x \\ & -0.35873 \cdot 10^{-5} x^{2} y^{3}+0.18891 \cdot 10^{-5} x^{2} y \\ & -0.58171 .10^{-5} x^{3} y+0.55752 \cdot 10^{-5} x y \\ & -0.98381 .10^{-5} x y^{2}+0.22139 \cdot 10^{-5} x y^{3} \\ & -2 x^{2} y^{2}-2 y^{2} z^{2}-2 x^{2} z^{2} \\ & -0.59775 .10^{-5} x^{2}+4 x^{2} y^{2} z^{2}+z^{4} \\ & +0.20153 .10^{-5} y^{2}+0.2 .10^{-4} I x^{2} z^{2} \\ & +0.2 .10^{-4} I x^{2} y^{2}+0.2 .10^{-4} I y^{2} z^{2} \\ & +0.42669 .10^{-5} x^{3}+x^{4}+y^{4} \end{aligned}$ |
| Bohemian dome | $\begin{aligned} & \frac{1-t^{2}}{1+t^{2}} \\ & \frac{1+2 t+t^{2}-s^{2}-s^{2} t^{2}+2 t s^{2}}{1+s^{2}+t^{2}+s^{2} t^{2}} \\ & \frac{2 s}{1+s^{2}} \end{aligned}$ | $\begin{aligned} & 2 x^{2} y^{2}-2 x^{2} z^{2}-4 y^{2} \\ & +x^{4}+z^{4}+2 y^{2} z^{2}+y^{4} \end{aligned}$ | $\begin{aligned} & 1.9999 x^{2} y^{2}+1.9999 y^{2} z^{2} \\ & +z^{4}+y^{4}-3.9999 y^{2} \\ & +x^{4}-1.9999 x^{2} z^{2} \end{aligned}$ |
| Swallowtail surface | $t s^{2}+3 s^{4},-2 t s-3 s^{3}, t$ | $\begin{aligned} & -15 x y^{2} z+3 y^{4}+y^{2} z^{3} \\ & -4 x z^{4}+12 x^{2} z^{2}-9 x^{3} \end{aligned}$ | $\begin{aligned} & -4 x z^{4}-10^{-4} I x^{3}+y^{2} z^{3} \\ & +10^{-4} I x z^{4}+12 x^{2} z^{2} \\ & -8.9999 x^{3}-15 x y^{2} z \\ & +10^{-4} I x^{2} z^{2}+2.9999 y^{4} \end{aligned}$ |
| Enneper's surface | $\begin{aligned} & t-\frac{t^{3}}{3}+t s^{2} \\ & 2-\frac{s^{3}}{3}+t^{2} s \\ & t^{2}-s^{2} \end{aligned}$ | $\begin{aligned} & 352836-78732 x^{2} y^{2} z+749412 z^{2}+ \\ & 101088 z^{3} x^{2} y-303264 x^{2} y z^{2}-25272 x^{2} y^{2} z^{3} \\ & -62127 z^{5}+75816 x^{2} y^{2} z^{2}+314928 x^{2} y z \\ & -4860 x^{4} z^{3}-2916 x^{6}+69984 x^{2} y^{3}+ \\ & 23328 y^{4} z^{2}-26244 y^{4} z+72576 y z^{5} \\ & +997272 y z-669222 y^{2} z-18144 y^{2} z^{5} \\ & +209952 y^{3} z-186624 y^{3} z^{2}+2592 x^{2} z^{5} \\ & -106920 x^{2} z^{3}+34992 x^{4}+5832 x^{4} z^{2} \\ & +8748 x^{4} y^{2}-34992 x^{4} y-183708 x^{2} y^{2} \\ & -268272 z^{4} y-1122660 z^{2} y+602640 z^{3} y \\ & +67068 z^{4} y^{2}-6912 z^{6} y+653913 z^{2} y^{2} \\ & +1728 z^{6} y^{2}-228420 z^{3} y^{2}+38880 z^{3} y^{3} \\ & -4860 z^{3} y^{4}-2304 z^{8}-536544 y^{3} \\ & +183708 y^{4}-34992 y^{5}+2916 y^{6} \\ & -577368 z-5616 z^{6}-8748 x^{2} y^{4}- \\ & -34992 x^{2}+2916 x^{2} z^{4}+305451 x^{2} z^{2} \\ & +7776 z^{7}-314928 x^{2} z+256 z^{9} \\ & -524151 z^{3}+916353 y^{2}+263898 z^{4} \\ & +174960 x^{2} y-866052 y-1728 x^{2} z^{6} \\ & \hline \end{aligned}$ |  |

Table 8: Parametric, implicit, and approximate implicit representation of surfaces; for the latter, we do not show coefficients of absolute value $<10^{-6}$.


[^0]:    *Email: \{emiris,kalinkat,thanglb\}@di.uoa.gr, ckonaxis@acmac.uoc.gr

[^1]:    ${ }^{1}$ http://sourceforge.net/projects/respol/files/

